

# Hamiltonian structure of an operator valued extension of Super KdV equations

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July 31, 2014

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## Abstract

An extension of the super Korteweg-de Vries integrable system in terms of operator valued functions is obtained. In particular the extension contains the  $N = 1$  Super KdV and coupled systems with functions valued on a symplectic space. We introduce a Miura transformation for the extended system and obtain its hamiltonian structure. We also obtain an extended Gardner transformation which allows to find an infinite number of conserved quantities of the extended system.

## 1 Introduction

The  $N = 1$  supersymmetric extension of Korteweg-de Vries equation obtained in [1] is equivalent to the equations obtained in [2] by reduction from the super Kadomtsev-Petviashvili hierarchy. The equations are described by a system of nonlinear coupled partial differential equations for fields which take values on the even and odd parts of a Grassmann algebra. The system posses a sequence of infinite local as well as non-local conserved quantities [1, 3, 4, 5]. Later on, also supersymmetric extensions with more than one supersymmetric generator were obtained [6, 7, 8, 9, 10].

Coupled partial differential equations in terms of real commuting fields may be obtained from these supersymmetric models by expanding the even and odd fields in terms of a basis of the Grassmann algebra and separating the equations corresponding to each

generator. Families of new solutions [11, 12] to super KdV were obtained using such bosonization procedure [13].

The bosonization approach becomes non-trivial when one considers the hamiltonian formulation, the Poisson structure on the unconstrained phase space and the Poisson structure on the constrained phase space given by the Dirac brackets. In fact, the formulation in terms of even variables introduces antisymmetric Poisson brackets while the one in terms of odd variables introduces symmetric Poisson brackets and the equivalence has to be shown explicitly. In the case we will consider in this paper both approaches becomes completely equivalent.

In this paper we will consider a very general algebraic structure for the fields describing the nonlinear systems. The systems are coupled partial differential equations extending the KdV equation. In particular our KdV extensions include the supersymmetric  $N = 1$  SKdV system as well as deformations of it. It also describe systems whose odd part satisfies a symplectic bracket structure.

We will obtain the hamiltonian structure of these extended KdV systems. The Poisson structure on the constrained submanifold of phase space will be given in terms of the Dirac brackets defined for constrained systems [14, 15, 16, 17]. The extended KdV systems we will consider have an infinite sequence of conserved quantities which may be obtained via a Gardner transformation as was done in [18, 13].

## 2 Extension of KdV equation, hamiltonian structure and infinite sequence of conserved quantities

We consider an associative algebra of even and odd elements. The even elements belong to a commutative algebra  $\mathcal{P}$  with unit while the odd ones belong to  $\mathcal{Q}$  which satisfies

$$\begin{aligned} \mathcal{Q}\mathcal{P} &\subset \mathcal{Q} \\ [\mathcal{Q}, \mathcal{P}] &= 0 \\ [\mathcal{Q}, \mathcal{Q}] &\subset \mathcal{P}, \end{aligned} \tag{1}$$

and for any  $q \in \mathcal{Q}$  there always exists  $\hat{q} \in \mathcal{Q}$  such that  $[q, \hat{q}] \neq 0$ .

We introduce a Lagrangian formulated in terms of fields  $w$  and  $\eta$  valued on  $\mathcal{P}$  and  $\mathcal{Q}$  respectively. The Lagrangian depends on a real parameter  $\lambda$ . The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{w}w' + \frac{\lambda}{2}[\dot{\eta}, \eta] - \frac{1}{2}(w'')^2 - \frac{1}{2}(w')^4 - \frac{1}{2}\lambda^2[\eta, \eta']^2 - \frac{\lambda}{2}[\eta'', \eta'] - \frac{3}{2}\lambda(w')^2[\eta', \eta]. \tag{2}$$

We notice that if  $\lambda$  is positive one may redefine  $\eta \rightarrow \lambda\eta$  and reduce the Lagrangian to the case  $\lambda = +1$ . If  $\lambda$  is negative one may reduce to the case  $\lambda = -1$ .

The associated field equations are

$$\dot{v} + v''' - 6v^2v' - 3\lambda(v[\eta', \eta])' = 0 \tag{3}$$

$$\dot{\eta} + \eta''' - 3v^2\eta' - 3vv'\eta + \lambda\eta'[\eta, \eta'] + \frac{1}{2}\eta\lambda[\eta, \eta''] = 0 \quad (4)$$

where  $v \equiv w'$ .

In particular if  $\mathcal{P}$  and  $\mathcal{Q}$  generates a Grassmann algebra, the latter two terms on the second equations are zero.

There is a hamiltonian and a corresponding Poisson structure associated to this Lagrangian.

We denote by  $p$  and  $\mu$  the conjugate momenta associated to  $w$  and  $\eta$  respectively.

We then obtain from the definition of  $p$  and  $\mu$  the following primary constraints

$$\begin{aligned} \phi &\equiv p - \frac{1}{2}v = 0 \\ \psi &\equiv \mu - \frac{\lambda}{2}\eta = 0. \end{aligned} \quad (5)$$

It turns out that these are the only constraints of the theory. They are second class constraints. In fact, they satisfy the following Poisson bracket relations

$$\begin{aligned} \{\phi(x), \phi(y)\}_{PB} &= -\partial_x \delta(x, y) \\ \{\phi(x), \psi(y)\}_{PB} &= 0 \\ \{\psi(x), \psi(y)\}_{PB} &= -\frac{1}{\lambda} \delta(x, y). \end{aligned}$$

The hamiltonian density associated to the Lagrangian (2) may be obtained via a Legendre transformation and is expressed as

$$\mathcal{H} = \frac{1}{2}(v')^2 + \frac{1}{2}v^4 + \frac{1}{2}\lambda^2[\eta, \eta']^2 + \frac{1}{2}\lambda[\eta'', \eta'] + \frac{3}{2}\lambda v^2[\eta', \eta] \quad (6)$$

subject to the constraints  $\phi = 0, \psi = 0$ .

The canonical field equations

$$\begin{aligned} \dot{v} &= \{v, H\}_{DB} \\ \dot{\eta} &= \{\eta, H\}_{DB} \end{aligned} \quad (7)$$

where  $H = \langle \mathcal{H} \rangle_x$  is the integral on  $\mathbb{R}$ , exactly agree with the Lagrangian field equations (3),(4) as it should be.

We notice that if  $\mathcal{P}$  and  $\mathcal{Q}$  generate a Grassmann algebra  $H$  reduces to the hamiltonian of the  $N = 1$  supersymmetric KdV equations.

In fact, in the case of a Grassmann algebra the term  $[\eta, \eta']^2$  in  $\mathcal{H}$  becomes zero and we may perform a Miura transformation

$$\begin{aligned} u &= v' + v^2 - \lambda[\eta, \eta'] \\ \xi &= \eta' + v\eta \end{aligned} \quad (8)$$

to obtain

$$\mathcal{H} = \frac{1}{2}u^2 + \frac{1}{2}\lambda[\xi', \xi] = \frac{1}{2}u^2 + \lambda\xi'\xi.$$

The canonical equations then reduce to the system

$$\begin{aligned} u_t &= -u''' + 6uu' - 6\lambda\xi\xi'' \\ \xi_t &= -\xi''' + 3(u\xi)', \end{aligned} \tag{9}$$

which is invariant under the supersymmetric transformation with odd parameter  $\epsilon$ ,

$$\begin{aligned} \delta_\epsilon u &= 2\epsilon\lambda\xi' \\ \delta_\epsilon \xi &= \epsilon u. \end{aligned} \tag{10}$$

(9) is a parametric Susy KdV equation, for  $\lambda = 1$  it gives the  $N = 1$  Super KdV equation.

If instead of considering a Grassmann algebra, we consider an associative algebra satisfying in addition to (1) the condition  $\mathcal{Q}\mathcal{Q} \subset \mathcal{P}$ , which implies  $[\mathcal{Q}, \mathcal{Q}] \subset \mathcal{P}$ , then the hamiltonian density reduces to

$$\mathcal{H} = \frac{1}{2}(v')^2 + \frac{1}{2}v^4 + \frac{1}{2}\lambda[\eta'', \eta'] + \frac{3}{2}\lambda v^2[\eta', \eta].$$

After using the generalized Miura transformation given by (8) we obtain

$$\mathcal{H} = \frac{1}{2}u^2 + \frac{\lambda}{2}[\xi', \xi]$$

and the canonical field equations

$$\begin{aligned} u_t &= -u''' + 6uu' + 3\lambda[\xi'', \xi] \\ \xi_t &= -\xi''' + 3(u\xi)', \end{aligned} \tag{11}$$

which are invariant under the transformations

$$\begin{aligned} \delta_\epsilon u &= \lambda[\epsilon, \xi'] \\ \delta_\epsilon \xi &= \epsilon u. \end{aligned} \tag{12}$$

Moreover, the system (11) under assumption (1) has an infinite sequence of local conserved quantities for any value of  $\lambda$ . This property may be proven by using a Gardner transformation as was done for the case  $\lambda = 1$  in [13]. In fact, we have

$$\begin{aligned} z_t &= (-z'' + 3z^2 + 3\lambda[\sigma', \sigma])' + \epsilon^2(2z^3 + 3\lambda z[\sigma', \sigma])' \\ \sigma_t &= (-\sigma'' + 3z\sigma)' + \epsilon^2 3(z^2\sigma' + z z'\sigma + \lambda\sigma'[\sigma', \sigma]), \end{aligned} \tag{13}$$

$$\begin{aligned} u &= z + \epsilon z' + \epsilon^2(z^2 + \lambda[\sigma', \sigma]) \\ \xi &= \sigma + \epsilon\sigma' + \epsilon^2 z\sigma, \end{aligned} \tag{14}$$

where (13) and (14) are the Gardner equations and associated Gardner transformations respectively.

After simplifying by crossing out derivatives in  $x$  and using the inverse Gardner transformation, the first four nontrivial conserved quantities for the operator-extended KdV system (11) are:

$$\begin{aligned}
H_0 &= \int u dx \\
H_2 &= \int (u^2 + \lambda [\xi', \xi]) dx \\
H_4 &= \int \left( 2u^3 + (u')^2 + 4\lambda u [\xi', \xi] + \lambda [\xi'', \xi'] \right) dx \\
H_6 &= \int \left( 5u^4 + 10u(u')^2 + (u'')^2 + 15\lambda u^2 [\xi', \xi] - 2\lambda u [\xi'', \xi'] \right. \\
&\quad \left. - 8\lambda u [\xi''', \xi] + 3\lambda^2 [\xi', \xi]^2 + \lambda [\xi''', \xi''] \right) dx.
\end{aligned} \tag{15}$$

### 3 Conclusions

We introduced an operatorial extension for Korteweg-de Vries equation which contains as particular cases several systems with the property of having a sequence of infinite local conserved quantities. In particular it contains the  $N = 1$  super KdV system. We obtained the hamiltonian structure of such extension. The existence of an infinite sequence of conserved quantities for the operatorial extension was shown using a generalized Gardner transformation.

#### Acknowledgments

A. R. and A. S. are partially supported by Project Fondecyt 1121103, Chile.

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